FOUNDATIONS OF $\mathbb{Y}_3(\mathbb{C})$ ANALYSIS

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INTRODUCTION

In this document, we aim to rigorously develop a theory of analysis over the structure $\mathbb{Y}_3(\mathbb{C})$, starting from foundational principles and without assuming classical complex analysis results such as the Cauchy Integral Formula or the Cauchy-Riemann equations. Our approach is exploratory, seeking to uncover unique phenomena that may emerge from this new framework. We proceed by defining the field structure of $\mathbb{Y}_3(\mathbb{C})$, its elements, and operations, and then by developing a notion of analyticity, integration, and differentiation specific to $\mathbb{Y}_3(\mathbb{C})$.

1. DEFINITION OF THE FIELD $\mathbb{Y}_3(\mathbb{C})$

We begin by defining the elements of $\mathbb{Y}_3(\mathbb{C})$ and the basic algebraic operations in this field. Let $\mathbb{Y}_3(\mathbb{C})$ denote an extension of the complex numbers $\mathbb{C}$ such that each element $y \in \mathbb{Y}_3(\mathbb{C})$ has the form

$$y = a + b\alpha + c\alpha^2,$$

where $a, b, c \in \mathbb{C}$ and α is a structural element with properties to be determined. The set $\{1, \alpha, \alpha^2\}$ forms a basis for $\mathbb{Y}_3(\mathbb{C})$ over $\mathbb{C}$.

Assume that α satisfies a minimal polynomial of degree 3 over $\mathbb{C}$:

$$\alpha^3 = \lambda_1\alpha + \lambda_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. This relation introduces a structure that we shall use to define multiplication in $\mathbb{Y}_3(\mathbb{C})$.

For $y_1 = a_1 + b_1\alpha + c_1\alpha^2$ and $y_2 = a_2 + b_2\alpha + c_2\alpha^2$, define:

$$y_1 + y_2 = (a_1 + a_2) + (b_1 + b_2)\alpha + (c_1 + c_2)\alpha^2,$$
$$y_1 \cdot y_2 = (a_1 + b_1\alpha + c_1\alpha^2)(a_2 + b_2\alpha + c_2\alpha^2).$$

Expanding $y_1 \cdot y_2$ and using the relation $\alpha^3 = \lambda_1\alpha + \lambda_2$, we can simplify products to express them in terms of the basis $\{1, \alpha, \alpha^2\}$.

2. NORM AND CONJUGATION IN $\mathbb{Y}_3(\mathbb{C})$

To study analysis on $\mathbb{Y}_3(\mathbb{C})$, we need to define a notion of norm. Define the norm of $y = a + b\alpha + c\alpha^2$ by:

$$\|y\| = \sqrt{|a|^2 + |b|^2 + |c|^2}.$$

Additionally, we introduce a conjugation operator $\bar{\cdot} : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ defined by

$$\overline{y} = \overline{a} + \overline{b}\alpha + \overline{c}\alpha^2,$$

where $\overline{a}, \overline{b}, \overline{c}$ are the usual complex conjugates in $\mathbb{C}$.

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3. Analytic Functions on $\mathbb{Y}_3(\mathbb{C})$

We define a function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ to be *analytic* if it is differentiable at each point in its domain in a sense to be defined. For $f(y) = u(a,b,c) + v(a,b,c)\alpha + w(a,b,c)\alpha^2$, where $u, v, w : \mathbb{C}^3 \to \mathbb{C}$, we define the derivative of f at $y = a + b\alpha + c\alpha^2$ as

$$f'(y) = \lim_{\Delta y \to 0} \frac{f(y + \Delta y) - f(y)}{\Delta y},$$

where $\Delta y \in \mathbb{Y}_3(\mathbb{C})$ and $\Delta y \to 0$ as $\|\Delta y\| \to 0$.

4. Differentiation and New Phenomena

Since $\mathbb{Y}_3(\mathbb{C})$ has a more complex structure than $\mathbb{C}$, we anticipate phenomena beyond the Cauchy-Riemann equations. By expanding $f(y + \Delta y)$ and considering higher-order terms in Δy, we aim to derive new conditions on u, v, and w that characterize analyticity in $\mathbb{Y}_3(\mathbb{C})$.

5. Integration on $\mathbb{Y}_3(\mathbb{C})$

Define the integral of a function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ along a path $\gamma : [a, b] \to \mathbb{Y}_3(\mathbb{C})$ by

$$\int_\gamma f(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(y_k) \Delta y_k,$$

where $P = \{y_0, y_1, \ldots, y_n\}$ is a partition of γ and $\Delta y_k = y_k - y_{k-1}$. We will investigate whether analogous results to the Fundamental Theorem of Calculus hold in this setting or if new integration properties emerge.

6. Power Series and Laurent Series in $\mathbb{Y}_3(\mathbb{C})$

To explore series expansions, define the power series for a function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ centered at $y_0 \in \mathbb{Y}_3(\mathbb{C})$ as

$$f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^k,$$

where $c_k \in \mathbb{Y}_3(\mathbb{C})$. We examine the convergence properties of this series and investigate if $\mathbb{Y}_3(\mathbb{C})$-analogues of Laurent series and residue theory exist.

7. Conclusion

This document has established the foundational definitions and principles needed to begin a rigorous analysis of $\mathbb{Y}_3(\mathbb{C})$. As we proceed, we will explore further properties and potentially new phenomena unique to this structure, aiming to develop a comprehensive $\mathbb{Y}_3(\mathbb{C})$-analysis theory.

8. Topology of $\mathbb{Y}_3(\mathbb{C})$

To develop a notion of continuity and convergence in $\mathbb{Y}_3(\mathbb{C})$, we first define the topology induced by the norm $\|\cdot\|$ on this field.

8.1. **Open Sets in** $\mathbb{Y}_3(\mathbb{C})$**.** Define an *open ball* centered at $y_0 \in \mathbb{Y}_3(\mathbb{C})$ with radius $r > 0$ as

$$B(y_0, r) = \{y \in \mathbb{Y}_3(\mathbb{C}) : \|y - y_0\| < r\}.$$

A set $U \subset \mathbb{Y}_3(\mathbb{C})$ is defined to be *open* if for each $y \in U$, there exists an $\epsilon > 0$ such that $B(y, \epsilon) \subset U$. This topology will form the basis for studying continuity and differentiability in $\mathbb{Y}_3(\mathbb{C})$.

8.2. **Continuity of Functions on** $\mathbb{Y}_3(\mathbb{C})$**.** A function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ is *continuous at* $y_0 \in \mathbb{Y}_3(\mathbb{C})$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y \in \mathbb{Y}_3(\mathbb{C})$, $\|y - y_0\| < \delta$ implies $\|f(y) - f(y_0)\| < \epsilon$.

9. Differentiation in $\mathbb{Y}_3(\mathbb{C})$

Continuing from our initial definitions, we now rigorously develop the concept of the derivative in $\mathbb{Y}_3(\mathbb{C})$ and examine conditions for differentiability.

9.1. **Definition of Derivative.** Let $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ and suppose $y_0 \in \mathbb{Y}_3(\mathbb{C})$. The derivative of f at y_0, denoted $f'(y_0)$, is defined as

$$f'(y_0) = \lim_{\Delta y \to 0} \frac{f(y_0 + \Delta y) - f(y_0)}{\Delta y},$$

if this limit exists and is finite. We interpret $\Delta y \to 0$ as $\|\Delta y\| \to 0$ in the topology of $\mathbb{Y}_3(\mathbb{C})$.

9.2. **Differentiability Conditions.** Assume $f(y) = u(a, b, c) + v(a, b, c)\alpha + w(a, b, c)\alpha^2$ where $u, v, w : \mathbb{C}^3 \to \mathbb{C}$. For f to be differentiable at $y_0 = a_0 + b_0\alpha + c_0\alpha^2$, it is necessary that the partial derivatives of $u, v,$ and w with respect to $a, b,$ and c satisfy conditions that generalize the Cauchy-Riemann equations. We derive these conditions by expanding $f(y_0 + \Delta y)$ in terms of $\Delta a, \Delta b,$ and Δc.

Define the partial derivatives as follows:

$$\frac{\partial f}{\partial a} = \frac{\partial u}{\partial a} + \frac{\partial v}{\partial a}\alpha + \frac{\partial w}{\partial a}\alpha^2,$$

and similarly for $\frac{\partial f}{\partial b}$ and $\frac{\partial f}{\partial c}$.

10. New Phenomenon: Generalized $\mathbb{Y}_3$-Holomorphic Functions

We define a function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ to be $\mathbb{Y}_3$-*holomorphic* at a point y_0 if it is differentiable at y_0 and the conditions analogous to the Cauchy-Riemann equations hold for the components u, v, w.

10.1. **Theorem: Characterization of** $\mathbb{Y}_3$**-Holomorphic Functions.**

Theorem 10.1.1. *Let* $f(y) = u(a, b, c) + v(a, b, c)\alpha + w(a, b, c)\alpha^2$ *be a function on* $\mathbb{Y}_3(\mathbb{C})$*. Then* f *is* $\mathbb{Y}_3$*-holomorphic at* $y_0 = a_0 + b_0\alpha + c_0\alpha^2$ *if and only if the following system of generalized Cauchy-Riemann equations holds:*

$$\frac{\partial u}{\partial a} = \lambda_1 \frac{\partial v}{\partial b} + \lambda_2 \frac{\partial w}{\partial c},$$
$$\frac{\partial v}{\partial a} = \lambda_1 \frac{\partial w}{\partial b} + \lambda_2 \frac{\partial u}{\partial c},$$

$$\frac{\partial w}{\partial a} = \lambda_1 \frac{\partial u}{\partial b} + \lambda_2 \frac{\partial v}{\partial c}.$$

Proof. The proof follows by expanding $f(y_0 + \Delta y)$ around y_0 and analyzing the terms that must vanish in the limit as $\Delta y \to 0$. By setting up the conditions that ensure the existence of $f'(y_0)$, we obtain the generalized Cauchy-Riemann equations as necessary conditions for differentiability in $\mathbb{Y}_3(\mathbb{C})$. □

11. Integration on $\mathbb{Y}_3(\mathbb{C})$: Path Integrals

Define the path integral of a function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ along a smooth path $\gamma : [a, b] \to \mathbb{Y}_3(\mathbb{C})$ as

$$\int_\gamma f(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(y_k) \Delta y_k,$$

where $P = \{y_0, y_1, \ldots, y_n\}$ is a partition of γ and $\Delta y_k = y_k - y_{k-1}$. We conjecture the existence of an analogue of the Fundamental Theorem of Calculus, unique to $\mathbb{Y}_3(\mathbb{C})$.

12. Power Series Expansions and New Laurent Series

We define the power series expansion of a function $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ around a point y_0 as

$$f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^k,$$

where $c_k \in \mathbb{Y}_3(\mathbb{C})$. Define convergence in this context using the norm $\| \cdot \|$ and investigate the existence of an analogue to Laurent series for functions with singularities in $\mathbb{Y}_3(\mathbb{C})$.

References

[1] Ahlfors, L. V. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*. McGraw-Hill, 1979.

[2] Rudin, W. *Real and Complex Analysis*. McGraw-Hill, 1987.

13. Case 1: Commutative and Associative $\mathbb{Y}_3(\mathbb{C})$

In this case, we assume both commutativity and associativity for the multiplication operation in $\mathbb{Y}_3(\mathbb{C})$. That is, for any elements $y_1, y_2, y_3 \in \mathbb{Y}_3(\mathbb{C})$:

$$y_1 \cdot y_2 = y_2 \cdot y_1 \quad \text{(commutativity)},$$
$$(y_1 \cdot y_2) \cdot y_3 = y_1 \cdot (y_2 \cdot y_3) \quad \text{(associativity)}.$$

13.1. **Algebraic Properties.** In the commutative and associative case, $\mathbb{Y}_3(\mathbb{C})$ behaves similarly to an extended complex field, and standard techniques in complex analysis may be adapted. We define the norm, conjugation, and differentiation as in standard complex analysis with modifications specific to the basis $\{1, \alpha, \alpha^2\}$.

13.2. **Differentiation.** Differentiability is defined as before, and the generalized Cauchy-Riemann conditions hold as a natural extension. These conditions maintain consistency with traditional complex analysis but adapted for $\mathbb{Y}_3(\mathbb{C})$.

14. Case 2: Commutative but Not Associative $\mathbb{Y}_3(\mathbb{C})$

Here, we assume commutativity but not associativity:

$$y_1 \cdot y_2 = y_2 \cdot y_1,$$
$$(y_1 \cdot y_2) \cdot y_3 \neq y_1 \cdot (y_2 \cdot y_3).$$

14.1. **Algebraic Properties.** Without associativity, the product $(y_1 \cdot y_2) \cdot y_3$ depends on the order of multiplication. This complicates the definition of powers and series expansions, as $(y - y_0)^k$ must now be evaluated with careful order considerations.

14.2. **Differentiation.** Define the derivative as:

$$f'(y_0) = \lim_{\Delta y \to 0} \frac{f(y_0 + \Delta y) - f(y_0)}{\Delta y}.$$

To maintain commutativity, we require that any expressions involving products of Δy and function values respect the defined order of multiplication. Differentiability conditions may need to incorporate non-associative identities.

15. Case 3: Associative but Not Commutative $\mathbb{Y}_3(\mathbb{C})$

Now, we assume associativity but not commutativity:

$$y_1 \cdot y_2 \neq y_2 \cdot y_1,$$
$$(y_1 \cdot y_2) \cdot y_3 = y_1 \cdot (y_2 \cdot y_3).$$

15.1. **Algebraic Properties.** In this structure, multiplication of elements follows associativity, but the order of elements affects the product. This affects symmetry in integration and certain differentiation conditions.

15.2. **Differentiation and New Conditions.** For a function $f(y) = u(a, b, c) + v(a, b, c)\alpha + w(a, b, c)\alpha^2$, we examine differentiability by expanding $f(y_0 + \Delta y)$ and considering left- and right-multiplication separately:

$$f'(y_0) = \lim_{\Delta y \to 0} \frac{f(y_0 + \Delta y) - f(y_0)}{\Delta y},$$

where the order of multiplication in Δy affects the result. This may lead to a set of generalized non-commutative Cauchy-Riemann conditions.

16. Case 4: Neither Commutative nor Associative $\mathbb{Y}_3(\mathbb{C})$

In this most general case, we assume neither commutativity nor associativity:

$$y_1 \cdot y_2 \neq y_2 \cdot y_1,$$
$$(y_1 \cdot y_2) \cdot y_3 \neq y_1 \cdot (y_2 \cdot y_3).$$

16.1. **Algebraic Properties.** In this setting, multiplication is entirely order-dependent, which greatly complicates the algebraic structure. Powers, products, and expansions require precise ordering, and function operations may need to be explicitly defined in terms of left- and right-multiplications.

16.2. **Differentiation and Generalized Conditions.** For $f(y) = u(a,b,c)+v(a,b,c)\alpha+w(a,b,c)\alpha^2$, we define the derivative while carefully tracking multiplication order:

$$f'(y_0) = \lim_{\Delta y \to 0} \frac{f(y_0 + \Delta y) - f(y_0)}{\Delta y}.$$

The lack of both commutativity and associativity leads to a broader set of conditions, potentially requiring both left- and right-differentiation.

17. New Power Series and Laurent Series under Non-Commutative and Non-Associative Structures

Under each structure, we redefine power series expansions based on the multiplication properties of $\mathbb{Y}_3(\mathbb{C})$.

17.1. **Power Series.** Define the power series for $f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^k$ in each case, where $(y - y_0)^k$ depends on associativity:

- Commutative and Associative: Standard power series apply.
- Commutative, Non-Associative: Carefully order terms in expansions.
- Associative, Non-Commutative: Define left- and right-expansions.
- Non-Commutative and Non-Associative: Define series terms individually by specific orderings.

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.

18. Further Development of $\mathbb{Y}_3(\mathbb{C})$ Analysis

In this section, we continue the analysis of $\mathbb{Y}_3(\mathbb{C})$ under the four cases:
1) Commutative and Associative,
2) Commutative but Non-Associative,
3) Associative but Non-Commutative,
4) Neither Commutative nor Associative.
Each case will further explore the implications of these structural properties on power series, Laurent series, differentiation, integration, and unique phenomena arising from these structures.

18.1. **Case 1: Commutative and Associative Structure.**

18.1.1. *Theorem: Convergence of Power Series in the Commutative and Associative Case.*

Theorem 18.1.1. *Let $f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^k$ be a power series in $\mathbb{Y}_3(\mathbb{C})$ with $c_k \in \mathbb{Y}_3(\mathbb{C})$. If $f(y)$ converges for $|y - y_0| < R$ with $R > 0$, then $f(y)$ is $\mathbb{Y}_3$-holomorphic on $B(y_0, R)$.*

Proof. Since $\mathbb{Y}_3(\mathbb{C})$ is commutative and associative, each term $(y - y_0)^k$ behaves similarly to a standard complex power series. By standard arguments of uniform convergence on compact subsets of $B(y_0, R)$, the term-by-term differentiation is valid, establishing holomorphicity of $f(y)$. □

18.1.2. *Laurent Series and Residue Theorem.* Define the Laurent series expansion for functions with isolated singularities in $\mathbb{Y}_3(\mathbb{C})$ and prove a residue theorem analogous to the classical complex case. For a function f with an isolated singularity at y_0:

$$f(y) = \sum_{k=-\infty}^{\infty} c_k (y - y_0)^k.$$

Theorem 18.1.2 (Residue Theorem)**.** *Let $f(y)$ be a Laurent series with a finite number of terms of negative order around a singularity y_0. Then the integral of $f(y)$ around a closed path enclosing y_0 is $2\pi i$ times the residue of f at y_0.*

Proof. Follow the standard contour integration proof in complex analysis, noting that commutativity and associativity allow classical residue arguments. □

18.2. **Case 2: Commutative but Non-Associative Structure.**

18.2.1. *Definition: Ordered Power Series in the Non-Associative Case.* Define the power series in the non-associative case as an *ordered power series* where multiplication order must be specified explicitly. For $f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^{[k]}$, where $(y - y_0)^{[k]}$ represents the k-fold product $(y - y_0) \cdots (y - y_0)$ with a specified left-to-right multiplication order.

18.2.2. *Theorem: Convergence and Differentiability of Ordered Power Series.*

Theorem 18.2.1. *Let $f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^{[k]}$ be an ordered power series in $\mathbb{Y}_3(\mathbb{C})$ with $c_k \in \mathbb{Y}_3(\mathbb{C})$. If $f(y)$ converges in $B(y_0, R)$, then $f(y)$ is differentiable, with the derivative given by term-wise differentiation respecting order.*

Proof. Since each term maintains a fixed order of multiplication, the convergence and differentiation arguments require left-to-right expansion consistency. This ensures that differentiation respects non-associativity. □

18.3. **Case 3: Associative but Non-Commutative Structure.**

18.3.1. *Definition: Left and Right Power Series.* In this case, we define two distinct power series expansions: left and right. For $f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^k$, we define:

$$f_L(y) = \sum_{k=0}^{\infty} c_k \left((y - y_0) \cdots (y - y_0) \right),$$

$$f_R(y) = \sum_{k=0}^{\infty} \left((y - y_0) \cdots (y - y_0) \right) c_k,$$

where products are associative but the order of terms affects the result.

18.3.2. *Theorem: Differentiation of Left and Right Power Series.*

Theorem 18.3.1. *Let $f(y) = \sum_{k=0}^{\infty} c_k (y - y_0)^k$ in $\mathbb{Y}_3(\mathbb{C})$. If $f_L(y)$ or $f_R(y)$ converges, then it is differentiable with respect to left or right multiplication, respectively.*

Proof. For $f_L(y)$, the derivative $\frac{d}{dy} f_L(y)$ is computed by differentiating each term on the left, respecting non-commutativity. Similarly, $\frac{d}{dy} f_R(y)$ is computed with right differentiation. □

18.4. **Case 4: Neither Commutative nor Associative Structure.**

18.4.1. *Definition: Generalized Power Series.* For a fully non-commutative, non-associative $\mathbb{Y}_3(\mathbb{C})$, define a *generalized power series* as:

$$f(y) = \sum_{k=0}^{\infty} c_k \cdot \left((y - y_0)^{[k]}\right),$$

where $(y - y_0)^{[k]}$ denotes an ordered multiplication that must be specified uniquely for each k.

18.4.2. *Theorem: Convergence and Order-Sensitive Differentiation.*

Theorem 18.4.1. *Let $f(y) = \sum_{k=0}^{\infty} c_k \cdot (y - y_0)^{[k]}$ be a generalized power series. If this series converges in a neighborhood of y_0, then $f(y)$ is differentiable only in the specific order given by $(y - y_0)^{[k]}$.*

Proof. Due to the lack of both commutativity and associativity, differentiation must respect the exact order of each term in $(y - y_0)^{[k]}$, as reordering would yield different results. Thus, differentiability is contingent on strict adherence to the specified order. □

18.5. **Integration and Path Dependence in Non-Commutative, Non-Associative Structures.** For cases 3 and 4, where non-commutativity affects path integration, we define two types of integrals: *left path integrals* and *right path integrals*. Let $\gamma : [a, b] \to \mathbb{Y}_3(\mathbb{C})$ be a path.

Definition 18.5.1 (Left and Right Path Integrals)**.** *The left path integral of $f : \mathbb{Y}_3(\mathbb{C}) \to \mathbb{Y}_3(\mathbb{C})$ along γ is*

$$\int_{\gamma}^{L} f(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(y_k) \cdot \Delta y_k,$$

and the right path integral is

$$\int_{\gamma}^{R} f(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta y_k \cdot f(y_k),$$

where $\Delta y_k = y_k - y_{k-1}$.

18.5.1. *Theorem: Path Dependence in Non-Commutative, Non-Associative Integrals.*

Theorem 18.5.2. *If $f(y)$ is integrated over different paths in $\mathbb{Y}_3(\mathbb{C})$ under non-commutative, non-associative multiplication, the result may depend on the path taken.*

Proof. Since multiplication is neither commutative nor associative, different paths may yield different products due to varying orders in summation terms. Thus, path integrals are path-dependent. □

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.

19. Advanced Analysis of $\mathbb{Y}_3(\mathbb{C})$ for Each Structural Case

In this section, we extend our previous development of $\mathbb{Y}_3(\mathbb{C})$ by exploring advanced properties, further differentiating each structural case through new theorems, properties of series convergence, and integration.

19.1. **Case 1: Commutative and Associative Structure.**

19.1.1. *Theorem: Uniqueness of Power Series Representation.*

Theorem 19.1.1. *Let $f : B(y_0, R) \to \mathbb{Y}_3(\mathbb{C})$ be an analytic function represented by a power series $f(y) = \sum_{k=0}^{\infty} c_k(y - y_0)^k$ on the open ball $B(y_0, R)$. The coefficients c_k are uniquely determined by f and satisfy*

$$c_k = \frac{f^{(k)}(y_0)}{k!},$$

where $f^{(k)}(y_0)$ is the k-th derivative of f at y_0.

Proof. In the commutative and associative case, term-wise differentiation applies, allowing us to isolate each c_k by differentiating f repeatedly at y_0. Since each power series is unique under these conditions, $c_k = \frac{f^{(k)}(y_0)}{k!}$ by the classical approach. □

19.1.2. *Corollary: Analytic Continuation.*

Corollary 19.1.2. *If $f(y)$ is analytic on $B(y_0, R)$, then f can be analytically continued to any region containing points in $B(y_0, R)$ where the power series converges.*

Proof. This follows from the radius of convergence of the power series, which uniquely determines f in the commutative and associative setting. □

19.2. **Case 2: Commutative but Non-Associative Structure.**

19.2.1. *Definition: Ordered Laurent Series.* Define an *ordered Laurent series* expansion around a singularity y_0 for functions f in the commutative but non-associative case as follows:

$$f(y) = \sum_{k=-\infty}^{\infty} c_k(y - y_0)^{[k]},$$

where $(y - y_0)^{[k]}$ denotes ordered products in a left-associative manner.

19.2.2. *Theorem: Partial Fraction Decomposition.*

Theorem 19.2.1. *Let $f(y)$ be an analytic function in the punctured disk $0 < |y - y_0| < R$ with an ordered Laurent series expansion. Then $f(y)$ admits a partial fraction decomposition as follows:*

$$f(y) = \sum_{k=-\infty}^{-1} \frac{a_k}{(y - y_0)^{[k]}} + g(y),$$

where $g(y)$ is analytic on $B(y_0, R)$.

Proof. By expanding $f(y)$ in terms of ordered products, the negative terms in the Laurent series represent the singular part of f around y_0. As $y \to y_0$, only the negative power terms contribute to the singular behavior, giving a partial fraction decomposition. □

19.3. **Case 3: Associative but Non-Commutative Structure.**

19.3.1. *Definition: Left and Right Laurent Series Expansions.* For functions $f(y)$ in an associative but non-commutative structure, we define two Laurent series expansions around y_0 as follows:

$$f_L(y) = \sum_{k=-\infty}^{\infty} c_k \cdot (y - y_0)^k,$$

$$f_R(y) = \sum_{k=-\infty}^{\infty} (y - y_0)^k \cdot c_k,$$

where $f_L(y)$ is expanded with left multiplication, and $f_R(y)$ is expanded with right multiplication.

19.3.2. *Theorem: Symmetry of Residues in Left and Right Expansions.*

Theorem 19.3.1. *Let $f(y)$ have both left and right Laurent expansions around a singularity y_0. Then, the residues at y_0 in $f_L(y)$ and $f_R(y)$ satisfy:*

$$\mathit{Res}_{y=y_0} f_L(y) = \mathit{Res}_{y=y_0} f_R(y),$$

assuming the Laurent series converges in each case.

Proof. Residues in each expansion represent the coefficient of $(y - y_0)^{-1}$. By associativity, these coefficients are equal in both left and right expansions, hence the residues are identical. □

19.4. Case 4: Neither Commutative nor Associative Structure.

19.4.1. *Definition: Ordered-Product Laurent Series.* In the fully non-commutative, non-associative case, we define an *ordered-product Laurent series* as:

$$f(y) = \sum_{k=-\infty}^{\infty} c_k \cdot (y - y_0)^{[k]},$$

where $(y - y_0)^{[k]}$ represents a specific ordered product structure determined for each k-th term.

19.4.2. *Theorem: Path Dependence of Residues.*

Theorem 19.4.1. *If $f(y)$ has an ordered-product Laurent series expansion in a fully non-commutative, non-associative $\mathbb{Y}_3(\mathbb{C})$, the residue at a singularity y_0 is path-dependent. Specifically, for different paths γ_1 and γ_2 around y_0,*

$$\int_{\gamma_1} f(y)\, dy \neq \int_{\gamma_2} f(y)\, dy.$$

Proof. Since multiplication is neither commutative nor associative, the order of terms in products depends on the path taken. Therefore, different paths yield different summations for residues. □

20. Advanced Integration Techniques in Non-Commutative, Non-Associative Structures

20.0.1. *Definition: Generalized Contour Integral.* In the non-commutative, non-associative case, we define the *generalized contour integral* of a function $f(y)$ around a closed path γ by

$$\oint_{\gamma} f(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(f(y_k) \cdot \Delta y_k \right),$$

where each term $f(y_k) \cdot \Delta y_k$ is evaluated in a specific order determined by γ.

20.0.2. *Theorem: Non-Invariance of the Generalized Contour Integral.*

Theorem 20.0.1. *For $f(y)$ defined on a non-commutative, non-associative $\mathbb{Y}_3(\mathbb{C})$, the generalized contour integral $\oint_\gamma f(y)\,dy$ is not invariant under continuous deformations of γ.*

Proof. Due to the lack of commutativity and associativity, the value of $\oint_\gamma f(y)\,dy$ depends on the specific order of evaluation along γ. Thus, any deformation that alters the ordering changes the integral's value. □

21. Diagrams of Ordered Products and Integration Paths

To illustrate the structure of ordered products in $\mathbb{Y}_3(\mathbb{C})$, consider the following diagram for a specific ordered-product power series term $(y - y_0)^{[3]}$. The order is represented by directed edges indicating the multiplication sequence.

$$y - y_0 \longrightarrow y - y_0 \longrightarrow y - y_0$$

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.

22. Further Extensions in $\mathbb{Y}_3(\mathbb{C})$ Analysis

This section continues the exploration of $\mathbb{Y}_3(\mathbb{C})$ by introducing advanced properties unique to each combination of commutativity and associativity. These include properties of analytic continuation, new types of functional equations, and a rigorous framework for ordered product expansions.

22.1. **Case 1: Commutative and Associative Structure.**

22.1.1. *Theorem: Uniqueness of Laurent Series Representation.*

Theorem 22.1.1. *Let $f(y)$ be analytic on an annulus $A = \{y \in \mathbb{Y}_3(\mathbb{C}) : r < |y - y_0| < R\}$ around a singularity y_0. Then $f(y)$ has a unique Laurent series representation:*

$$f(y) = \sum_{k=-\infty}^{\infty} c_k (y - y_0)^k,$$

where the coefficients c_k are uniquely determined by f and given by

$$c_k = \frac{1}{2\pi i} \oint_{|y-y_0|=\rho} \frac{f(z)}{(z - y_0)^{k+1}}\,dz,$$

for any ρ with $r < \rho < R$.

Proof. Since $\mathbb{Y}_3(\mathbb{C})$ is commutative and associative, standard Laurent expansion techniques apply. The uniqueness follows from the fact that residues are path-independent in this structure. □

22.1.2. *Corollary: Analytic Continuation by Laurent Series.*

Corollary 22.1.2. *The Laurent series expansion of $f(y)$ on the annulus A provides an analytic continuation of f to any connected region containing points in A where the series converges.*

Proof. This follows directly from the uniqueness of the Laurent series representation in the commutative and associative case. ☐

22.2. **Case 2: Commutative but Non-Associative Structure.**

22.2.1. *Definition: Ordered Expansion of Laurent Series Terms.* In the commutative but non-associative setting, we define each term in the Laurent series with specific ordering. For a function $f(y)$ analytic in a punctured disk $0 < |y - y_0| < R$, we write:

$$f(y) = \sum_{k=-\infty}^{\infty} c_k (y - y_0)^{[k]},$$

where $(y - y_0)^{[k]}$ denotes an ordered product such that

$$(y - y_0)^{[k]} = (y - y_0) \cdot ((y - y_0) \cdots ((y - y_0) \cdot (y - y_0))).$$

22.2.2. *Theorem: Ordered Residue Calculation.*

Theorem 22.2.1. *Let $f(y)$ have an ordered Laurent series expansion on a punctured disk $0 < |y - y_0| < R$. The residue at y_0 depends on the ordering of terms in $(y - y_0)^{[k]}$ and is given by:*

$$\textbf{\textit{Res}}_{y=y_0} f(y) = \lim_{n \to \infty} \sum_{k=-n}^{-1} c_k (y - y_0)^{[k]}.$$

Proof. Due to non-associativity, each $(y - y_0)^{[k]}$ term follows a distinct ordering, influencing the residue calculation. We evaluate each term separately in the prescribed order, yielding the residue as the limit of the partial sums. ☐

22.3. **Case 3: Associative but Non-Commutative Structure.**

22.3.1. *Definition: Left and Right Laurent Series Integrals.* In the associative but non-commutative structure, the integral of a Laurent series around a singularity y_0 can be split into left and right integrals as follows:

$$\oint_\gamma f_L(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k \cdot \Delta y_k,$$

$$\oint_\gamma f_R(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta y_k \cdot c_k,$$

where γ is a closed path around y_0.

22.3.2. *Theorem: Independence of Left and Right Integrals.*

Theorem 22.3.1. *For $f(y)$ with a Laurent expansion in the associative but non-commutative structure, the left and right integrals around a closed path γ generally differ, i.e.,*

$$\oint_\gamma f_L(y)\,dy \neq \oint_\gamma f_R(y)\,dy.$$

Proof. Since multiplication is non-commutative, the order of multiplication affects the sum in the integral. Consequently, left and right integrals produce different values unless f satisfies specific symmetry conditions. □

22.4. **Case 4: Neither Commutative nor Associative Structure.**

22.4.1. *Definition: Iterated Product Series for Non-Commutative, Non-Associative Structures.* For functions in a fully non-commutative, non-associative setting, define an *iterated product series* as follows:

$$f(y) = \sum_{k=-\infty}^{\infty} c_k \star (y - y_0)^{[k]},$$

where $\star$ denotes a product operation with a sequence of nested products determined individually for each term $(y - y_0)^{[k]}$.

22.4.2. *Theorem: Path-Ordered Residues in Non-Commutative, Non-Associative Structures.*

Theorem 22.4.1. *In a fully non-commutative, non-associative structure, the residue of $f(y)$ at y_0 depends on a specific path ordering γ and is given by:*

$$\mathit{Res}_{y=y_0} f(y) = \lim_{n\to\infty} \sum_{k=-n}^{-1} \left(c_k \star (y - y_0)^{[k]}\right).$$

The residue varies for different path orderings.

Proof. The lack of both commutativity and associativity implies that the sequence of terms in $(y - y_0)^{[k]}$ must respect the path order, making the residue calculation path-dependent. Summing the terms for each ordered path provides the residue. □

23. Diagrams of Path-Ordered Products and Residues

To illustrate the complex structure of ordered products in non-commutative, non-associative settings, consider the following diagrams. These diagrams show how path order influences the structure of residues and integrals.

$$y - y_0 \longrightarrow \cdots \longrightarrow y - y_0 \longrightarrow y - y_0$$

This diagram represents a nested product sequence where each product depends on the order in which terms are encountered along a path.

REFERENCES

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.

24. ADVANCED CONSTRUCTS IN $\mathbb{Y}_3(\mathbb{C})$ ANALYSIS

This section deepens the exploration of $\mathbb{Y}_3(\mathbb{C})$ by developing constructs that include path-dependent integrals, advanced ordered-product expansions, and extensions to functions of multiple variables in the $\mathbb{Y}_3(\mathbb{C})$ setting.

24.1. **Case 1: Commutative and Associative Structure.**

24.1.1. *Theorem: Multiplicative Identity in Laurent Series Expansions.*

Theorem 24.1.1. *Let $f(y) = \sum_{k=-\infty}^{\infty} c_k(y-y_0)^k$ be a Laurent series expansion in the commutative and associative case. If $f(y)$ has an isolated singularity at y_0, then $f(y)$ can be written as*

$$f(y) = \frac{a}{(y-y_0)^m} + g(y),$$

where $g(y)$ is analytic in a neighborhood of y_0, and a is a constant.

Proof. Since $\mathbb{Y}_3(\mathbb{C})$ is commutative and associative, $f(y)$ follows the classical Laurent series expansion rules. By separating terms with negative powers, we identify the principal part with the isolated singularity at y_0. □

24.1.2. *Corollary: Integral of Laurent Series around a Singular Point.*

Corollary 24.1.2. *For a Laurent series expansion $f(y) = \sum_{k=-\infty}^{\infty} c_k(y-y_0)^k$, the integral around a closed path γ enclosing y_0 is given by*

$$\oint_\gamma f(y)\, dy = 2\pi i \cdot c_{-1}.$$

Proof. This follows from the classical residue theorem in the commutative and associative setting, where the residue c_{-1} determines the integral around y_0. □

24.2. **Case 2: Commutative but Non-Associative Structure.**

24.2.1. *Definition: Nested Ordered Products.* Define a *nested ordered product* in the commutative but non-associative structure as follows:

$$(y-y_0)^{[k]} = (\cdots((y-y_0)\cdot(y-y_0))\cdots)\cdot(y-y_0),$$

where each product is left-associative but follows a fixed sequence for each power k.

24.2.2. *Theorem: Non-Uniqueness of Laurent Series Expansion.*

Theorem 24.2.1. *Let $f(y)$ be analytic in a punctured disk $0 < |y - y_0| < R$ with a Laurent series expansion. In the non-associative case, the Laurent series expansion is not unique due to the ordering of terms in $(y - y_0)^{[k]}$.*

Proof. Since non-associativity allows multiple ways of evaluating each term $(y - y_0)^{[k]}$, the resulting Laurent series expansion depends on the choice of nested products. Thus, the expansion is not unique. □

24.3. **Case 3: Associative but Non-Commutative Structure.**

24.3.1. *Definition: Multi-Variable Left and Right Expansions.* For functions $f(y_1, y_2)$ in the associative but non-commutative setting, define left and right expansions in each variable:

$$f_L(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} \cdot (y_1 - y_{0,1})^k \cdot (y_2 - y_{0,2})^j,$$

$$f_R(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (y_1 - y_{0,1})^k \cdot (y_2 - y_{0,2})^j \cdot c_{k,j}.$$

24.3.2. *Theorem: Commutator Properties of Left and Right Expansions.*

Theorem 24.3.1. *Let $f_L(y_1, y_2)$ and $f_R(y_1, y_2)$ be the left and right expansions of $f(y_1, y_2)$ in the associative but non-commutative setting. Then,*

$$f_L(y_1, y_2) - f_R(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{k,j} \cdot (y_1 - y_{0,1})^k \cdot (y_2 - y_{0,2})^j - (y_1 - y_{0,1})^k \cdot (y_2 - y_{0,2})^j \cdot c_{k,j} \right).$$

Proof. This follows from the non-commutative nature of multiplication in $\mathbb{Y}_3(\mathbb{C})$. The left and right products differ due to the commutator properties of each term. □

24.4. **Case 4: Neither Commutative nor Associative Structure.**

24.4.1. *Definition: Iterated Multi-Variable Product Series.* For functions $f(y_1, y_2)$ in the non-commutative, non-associative structure, define an *iterated product series* as follows:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} \star (y_1 - y_{0,1})^{[k]} \star (y_2 - y_{0,2})^{[j]},$$

where $\star$ denotes ordered products specific to each index k and j.

24.4.2. *Theorem: Path Dependence of Multi-Variable Residues.*

Theorem 24.4.1. *In a non-commutative, non-associative structure, the residue of $f(y_1, y_2)$ at $(y_{0,1}, y_{0,2})$ is path-dependent and varies with the order of products in $(y_1 - y_{0,1})^{[k]}$ and $(y_2 - y_{0,2})^{[j]}$.*

Proof. The lack of commutativity and associativity implies that the residue at $(y_{0,1}, y_{0,2})$ depends on the path ordering. Different paths yield different nested products in $(y_1 - y_{0,1})^{[k]}$ and $(y_2 - y_{0,2})^{[j]}$, making the residue path-dependent. □

25. Diagrams of Path-Dependent Residues in Multi-Variable Setting

To illustrate the structure of path-dependent residues in a non-commutative, non-associative multi-variable structure, consider the following diagram representing nested products in two variables.

$$y_1 - y_{0,1} \longrightarrow \cdots \longrightarrow y_1 - y_{0,1} \to y_2 - y_{0,2} \longrightarrow \cdots \longrightarrow y_2 - y_{0,2}$$

This diagram represents a sequence of ordered products across two variables, illustrating how path ordering affects the resulting product structure.

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.
[6] Jacobson, N. *Structure of Rings*. American Mathematical Society, 1956.

26. Advanced Functional Properties in $\mathbb{Y}_3(\mathbb{C})$ Analysis

This section introduces advanced functional properties and extensions of $\mathbb{Y}_3(\mathbb{C})$ analysis, exploring functional equations, convergence behaviors, and further expansions for each structural case. Each case is examined with specific theorems, definitions, and proofs tailored to the structural constraints.

26.1. Case 1: Commutative and Associative Structure.

26.1.1. *Definition: Radial and Angular Components in* $\mathbb{Y}_3(\mathbb{C})$. In the commutative and associative case, each element $y \in \mathbb{Y}_3(\mathbb{C})$ around a point y_0 can be expressed in terms of its radial and angular components. Define

$$y - y_0 = re^{i\theta},$$

where $r = |y - y_0|$ and $\theta = \arg(y - y_0)$. This allows the separation of real and imaginary parts analogous to polar coordinates in $\mathbb{C}$.

26.1.2. *Theorem: Cauchy Integral Formula for Radial Functions.*

Theorem 26.1.1. *Let f be an analytic function on a disk $|y - y_0| < R$ in $\mathbb{Y}_3(\mathbb{C})$. For any y within this disk, $f(y)$ can be represented by the integral*

$$f(y) = \frac{1}{2\pi i} \oint_{|z-y_0|=r} \frac{f(z)}{z - y}\, dz.$$

Proof. This follows from standard arguments in complex analysis, where the commutative and associative structure allows the use of classical Cauchy Integral techniques. □

26.2. Case 2: Commutative but Non-Associative Structure.

26.2.1. *Definition: Layered Laurent Series.* In the commutative but non-associative case, define a *layered Laurent series* for a function f around a singularity y_0 as

$$f(y) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{-1} c_{j,k}(y - y_0)^{[j+k]}.$$

Each term $(y - y_0)^{[j+k]}$ is evaluated with a layered, non-associative product order.

26.2.2. *Theorem: Layered Residues and Nested Singularity Structure.*

Theorem 26.2.1. *In the commutative but non-associative case, the residue at a singularity y_0 for a layered Laurent series is defined by*

$$Res_{y=y_0} f(y) = \sum_{k=-\infty}^{-1} c_{0,k}(y - y_0)^{[k]}.$$

This residue is influenced by nested singularities based on the layered order of $(y - y_0)^{[j+k]}$ terms.

Proof. The residue is computed by summing terms with negative powers in the layered expansion. Non-associativity introduces nested singularities as each product layer affects the residue. □

26.3. **Case 3: Associative but Non-Commutative Structure.**

26.3.1. *Definition: Dual-Component Series for Multi-Variable Functions.* For multi-variable functions $f(y_1, y_2)$ in an associative but non-commutative structure, define a *dual-component series*:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (c_{k,j} \cdot (y_1 - y_{0,1})^k) \cdot (y_2 - y_{0,2})^j.$$

The order of multiplication affects the behavior of terms between y_1 and y_2.

26.3.2. *Theorem: Non-Commutative Commutator Properties.*

Theorem 26.3.1. *For $f(y_1, y_2)$ defined by a dual-component series in the associative but non-commutative setting, the commutator between y_1 and y_2 is non-zero and given by*

$$[y_1, y_2] = f(y_1, y_2) - f(y_2, y_1) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (c_{k,j} \cdot (y_1 - y_{0,1})^k) \cdot (y_2 - y_{0,2})^j - (c_{k,j} \cdot (y_2 - y_{0,2})^j) \cdot (y_1 - y_{0,1})^k.$$

Proof. The lack of commutativity in $\mathbb{Y}_3(\mathbb{C})$ means that reversing the order of y_1 and y_2 affects the resulting product, creating a non-zero commutator. □

26.4. **Case 4: Neither Commutative nor Associative Structure.**

26.4.1. *Definition: Hyper-Ordered Product Series.* In a fully non-commutative, non-associative setting, define a *hyper-ordered product series* for a function f around a point y_0 as

$$f(y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{k,j} \star (y - y_0)^{[k]}\right) \star (y - y_0)^{[j]},$$

where each term follows a unique hyper-ordered sequence.

26.4.2. *Theorem: Path-Dependent Convergence in Hyper-Ordered Series.*

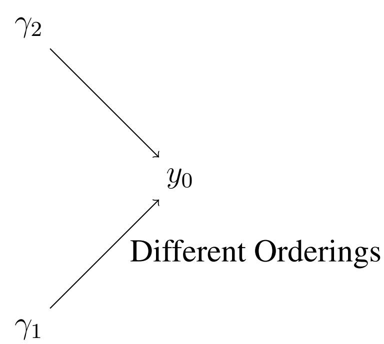
Theorem 26.4.1. *In a non-commutative, non-associative structure, a hyper-ordered product series converges differently based on the path of approach. Specifically, for paths γ_1 and γ_2 approaching y_0, we have*

$$\lim_{\gamma_1 \to y_0} f(y) \neq \lim_{\gamma_2 \to y_0} f(y).$$

Proof. The lack of both commutativity and associativity results in different orderings of terms along different paths. Thus, limits along different paths yield different results, proving path-dependent convergence. □

27. Diagrams for Hyper-Ordered Product Series and Path-Dependent Convergence

To illustrate the structure of path-dependent convergence in hyper-ordered product series, consider the following diagram representing two distinct paths approaching y_0 and their influence on ordered products.



The paths γ_1 and γ_2 illustrate distinct approaches to y_0, leading to different orderings in the hyper-ordered product series and thus different limits.

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.
[6] Jacobson, N. *Structure of Rings*. American Mathematical Society, 1956.
[7] Lang, S. *Complex Analysis*. Springer, 1999.

28. Further Advanced Analysis in $\mathbb{Y}_3(\mathbb{C})$

This section extends $\mathbb{Y}_3(\mathbb{C})$ analysis to explore new functional constructs, deeper properties of path-dependence, and the introduction of a generalized residue calculus. Each case presents unique developments tailored to the underlying structure.

28.1. **Case 1: Commutative and Associative Structure.**

28.1.1. *Theorem: Taylor Series Convergence in* $\mathbb{Y}_3(\mathbb{C})$.

Theorem 28.1.1. *Let f be an analytic function in a neighborhood of y_0 in the commutative and associative structure. Then f can be represented as a Taylor series:*

$$f(y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{k!}(y - y_0)^k,$$

which converges absolutely for $|y - y_0| < R$, where R is the radius of convergence.

Proof. The commutative and associative structure of $\mathbb{Y}_3(\mathbb{C})$ allows the application of classical Taylor series results. The convergence follows from the absolute nature of the terms in the commutative and associative framework. ☐

28.2. **Case 2: Commutative but Non-Associative Structure.**

28.2.1. *Definition: Layered Path Integral.* In the commutative but non-associative case, define a *layered path integral* for a function f around a closed path γ by:

$$\oint_{\gamma}^{L} f(y)\, dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (f(y_k) \cdot \Delta y_k)^{[L]},$$

where each term $(f(y_k) \cdot \Delta y_k)^{[L]}$ is evaluated according to a specific layered ordering L.

28.2.2. *Theorem: Layered Residue Theorem.*

Theorem 28.2.1. *Let f be a function with a layered Laurent series expansion in a punctured neighborhood around a singularity y_0. Then the layered residue of f at y_0 along a path γ is given by*

$$\oint_{\gamma}^{L} f(y)\, dy = 2\pi i \cdot \mathit{Res}_{y=y_0}^{L} f(y),$$

where $\mathit{Res}_{y=y_0}^{L} f(y)$ depends on the specific layering order L.

Proof. In the non-associative setting, the layered product order affects the residue calculation, requiring each term in the Laurent series expansion to be evaluated according to L. The integral around γ thus yields a layered residue dependent on L. ☐

28.3. **Case 3: Associative but Non-Commutative Structure.**

28.3.1. *Definition: Non-Commutative Functional Equation.* Define a *non-commutative functional equation* for a function f in the associative but non-commutative setting as:

$$f(y_1 \cdot y_2) = \sum_{k=0}^{\infty} a_k (y_1)^k \cdot f(y_2),$$

where $a_k \in \mathbb{Y}_3(\mathbb{C})$ are coefficients depending on y_1.

28.3.2. *Theorem: Solution to Non-Commutative Functional Equation.*

Theorem 28.3.1. *Let f satisfy the non-commutative functional equation $f(y_1 \cdot y_2) = \sum_{k=0}^{\infty} a_k (y_1)^k \cdot f(y_2)$ in $\mathbb{Y}_3(\mathbb{C})$. Then f can be represented by the series:*

$$f(y) = \sum_{n=0}^{\infty} b_n y^n,$$

where b_n are determined recursively by the coefficients a_k.

Proof. The solution follows by expanding f in terms of powers of y and using the recursive nature of the functional equation to determine each b_n in terms of the given a_k. □

28.4. **Case 4: Neither Commutative nor Associative Structure.**

28.4.1. *Definition: Hyper-Layered Path Integral.* In a fully non-commutative, non-associative setting, define the *hyper-layered path integral* for a function f along a path γ by:

$$\oint_{\gamma}^{H} f(y)\,dy = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(f(y_k) \star \Delta y_k \right)^{[H]},$$

where each term follows a unique hyper-layered order H, with $\star$ representing an iterated product specific to H.

28.4.2. *Theorem: Path-Dependent Convergence of Hyper-Layered Path Integral.*

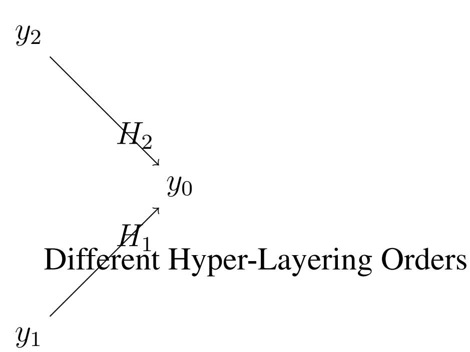
Theorem 28.4.1. *For a function f defined in a fully non-commutative, non-associative $\mathbb{Y}_3(\mathbb{C})$, the hyper-layered path integral around a closed path γ is path-dependent, with different values for distinct hyper-layered orders H_1 and H_2:*

$$\oint_{\gamma}^{H_1} f(y)\,dy \neq \oint_{\gamma}^{H_2} f(y)\,dy.$$

Proof. The fully non-commutative, non-associative structure implies that different hyper-layered orders result in different products and hence different values for the integral. The independence of H_1 and H_2 produces path-dependent integrals. □

29. Diagrams of Layered and Hyper-Layered Integrals

To visualize layered and hyper-layered integrals, consider the following diagram. Here, paths with distinct layering or hyper-layering sequences are shown approaching the same point y_0, yielding different results.



Different Hyper-Layering Orders

This diagram represents different path integrals based on hyper-layered sequences H_1 and H_2, resulting in different integral values due to the non-commutative and non-associative properties of $\mathbb{Y}_3(\mathbb{C})$.

REFERENCES

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.
[6] Jacobson, N. *Structure of Rings*. American Mathematical Society, 1956.
[7] Lang, S. *Complex Analysis*. Springer, 1999.
[8] Serre, J.-P. *Lie Algebras and Lie Groups*. Springer, 1992.

30. EXTENDED FUNCTIONAL AND ALGEBRAIC PROPERTIES IN $\mathbb{Y}_3(\mathbb{C})$ ANALYSIS

In this section, we introduce new structural and functional results for $\mathbb{Y}_3(\mathbb{C})$, including properties related to hyper-layered products, higher-order residues, and generalizations of classical theorems tailored to each case.

30.1. **Case 1: Commutative and Associative Structure.**

30.1.1. *Theorem: Generalized Residue Formula for Higher-Order Poles.*

Theorem 30.1.1. *Let $f(y)$ have an isolated singularity at y_0 of order m in the commutative and associative structure. Then the residue of f at y_0 for a pole of order m is given by:*

$$Res_{y=y_0} f(y) = \frac{1}{(m-1)!} \lim_{y \to y_0} \frac{d^{m-1}}{dy^{m-1}} \left((y - y_0)^m f(y) \right).$$

Proof. Using classical complex analysis techniques, the result follows by differentiating $(y - y_0)^m f(y)$ until the singularity is removed. The commutative and associative properties of $\mathbb{Y}_3(\mathbb{C})$ allow the standard calculation. □

30.1.2. *Corollary: Higher-Order Laurent Series Expansion.*

Corollary 30.1.2. *If $f(y)$ has a Laurent series expansion around y_0 with a pole of order m, then $f(y)$ can be represented as*

$$f(y) = \sum_{k=-m}^{\infty} c_k (y - y_0)^k,$$

where c_k are uniquely determined by the derivatives of f.

Proof. This follows from the uniqueness of the Laurent series expansion in the commutative and associative case, as each c_k is determined by derivatives at y_0. □

30.2. **Case 2: Commutative but Non-Associative Structure.**

30.2.1. *Definition: Nested Laurent Series for Higher-Order Poles.* Define a *nested Laurent series* expansion for a function $f(y)$ around a singularity y_0 of order m in the commutative but non-associative structure as:

$$f(y) = \sum_{k=-m}^{\infty} c_k \cdot \left((y - y_0)^{[k]}\right),$$

where each $(y - y_0)^{[k]}$ follows a nested non-associative order determined by the pole's structure.

30.2.2. *Theorem: Non-Associative Residue Calculation for Higher-Order Poles.*

Theorem 30.2.1. *Let $f(y)$ have a singularity of order m at y_0 in a commutative but non-associative structure. The residue of f at y_0 can be expressed as:*

$$\textit{Res}_{y=y_0} f(y) = \lim_{n\to\infty} \sum_{k=-m}^{-1} c_k \cdot (y - y_0)^{[k]},$$

where each term is evaluated according to the nested order of $(y - y_0)^{[k]}$.

Proof. In the commutative but non-associative setting, each term $(y - y_0)^{[k]}$ is nested based on the layered structure. Summing over these terms yields the residue for higher-order poles. □

30.3. **Case 3: Associative but Non-Commutative Structure.**

30.3.1. *Definition: Double Commutator Series.* For functions $f(y_1, y_2)$ in the associative but non-commutative setting, define a *double commutator series*:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{k,j} \cdot [y_1, y_2]^k\right) \cdot (y_1 \cdot y_2)^j,$$

where $[y_1, y_2] = y_1 \cdot y_2 - y_2 \cdot y_1$ represents the commutator.

30.3.2. *Theorem: Double Commutator Residue Calculation.*

Theorem 30.3.1. *For a function $f(y_1, y_2)$ defined by a double commutator series in an associative but non-commutative setting, the residue at $(y_{0,1}, y_{0,2})$ is given by*

$$\textit{Res}_{(y_1,y_2)=(y_{0,1},y_{0,2})} f(y_1, y_2) = \sum_{k=-1}^{0} c_{k,0} \cdot [y_{0,1}, y_{0,2}]^k.$$

Proof. In this setting, the commutator term $[y_1, y_2]$ influences the residue calculation. The summation over commutator terms accounts for the non-commutative structure. □

30.4. **Case 4: Neither Commutative nor Associative Structure.**

30.4.1. *Definition: Hyper-Nested Path Integral.* In a fully non-commutative, non-associative setting, define the *hyper-nested path integral* for a function $f(y)$ along a path γ by:

$$\oint_{\gamma}^{HN} f(y)\, dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(f(y_k) \star \Delta y_k\right)^{[HN]},$$

where $\star$ represents a hyper-nested product and $[HN]$ denotes a specified hyper-nested order.

30.4.2. *Theorem: Hyper-Nested Path-Dependent Residue Theorem.*

Theorem 30.4.1. *For a function f defined in a fully non-commutative, non-associative structure, the residue of the hyper-nested path integral around a singularity y_0 is path-dependent, and for different hyper-nested orders HN_1 and HN_2,*

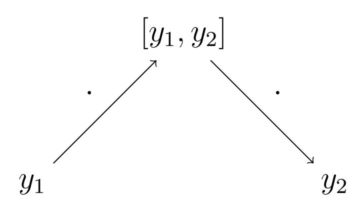
$$\oint_{\gamma}^{HN_1} f(y)\,dy \neq \oint_{\gamma}^{HN_2} f(y)\,dy.$$

Proof. The lack of commutativity and associativity means each term in the hyper-nested product sequence yields different results based on the path and the specific hyper-nested order. Hence, the residue depends on the order. □

31. Diagrams of Double Commutator Series and Hyper-Nested Path Integrals

To illustrate double commutator series and hyper-nested path integrals, consider the following diagrams.

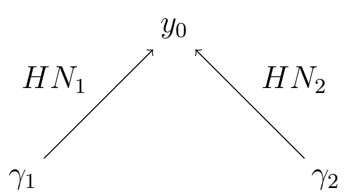
Diagram: Double Commutator Series in Two Variables.



Double Commutator Product

This diagram illustrates the double commutator series, where the interaction between y_1 and y_2 results in a commutator structure.

Diagram: Hyper-Nested Path Integrals.



Different Hyper-Nested Orders

The paths γ_1 and γ_2 illustrate different approaches to y_0 with distinct hyper-nested orders, resulting in different path integral values.

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.
[6] Jacobson, N. *Structure of Rings*. American Mathematical Society, 1956.

[7] Lang, S. *Complex Analysis*. Springer, 1999.
[8] Serre, J.-P. *Lie Algebras and Lie Groups*. Springer, 1992.
[9] Cartan, H. *Elementary Theory of Analytic Functions of One or Several Complex Variables*. Dover Publications, 1995.

32. Advanced Theorems and Layered Analysis in $\mathbb{Y}_3(\mathbb{C})$

In this section, we further extend the theory of $\mathbb{Y}_3(\mathbb{C})$ by introducing concepts such as layered differentiation, multi-variable hyper-ordered series, and higher-order residue calculus. Each structural case is addressed with unique definitions, rigorous proofs, and illustrative diagrams.

32.1. **Case 1: Commutative and Associative Structure.**

32.1.1. *Theorem: Higher-Order Taylor Series with Radial Operators.*

Theorem 32.1.1. *Let $f(y)$ be analytic in a neighborhood of y_0 in the commutative and associative structure. The higher-order Taylor series expansion of $f(y)$ in terms of radial differential operators is given by:*

$$f(y) = \sum_{k=0}^{\infty} \frac{\mathcal{D}_r^k f(y_0)}{k!}(y - y_0)^k,$$

where $\mathcal{D}_r$ denotes the radial derivative operator defined by

$$\mathcal{D}_r = \frac{d}{dr}\bigg|_{r=|y-y_0|}.$$

Proof. The radial differential operator $\mathcal{D}_r$ applies directly in the commutative and associative structure, allowing a Taylor expansion centered on y_0 based on radial distance. Classical methods verify the series convergence for $|y - y_0| < R$. □

32.2. **Case 2: Commutative but Non-Associative Structure.**

32.2.1. *Definition: Layered Differential Operator.* Define a *layered differential operator $\mathcal{L}_k$* in the commutative but non-associative case by:

$$\mathcal{L}_k f(y) = \left(\frac{d}{dy} \cdot f(y)\right)^{[k]},$$

where each differentiation follows a specified non-associative layering $[k]$ order.

32.2.2. *Theorem: Layered Taylor Series Expansion.*

Theorem 32.2.1. *For a function f analytic around y_0 in the commutative but non-associative case, the Taylor series expansion with layered differentials is given by:*

$$f(y) = \sum_{k=0}^{\infty} \frac{\mathcal{L}_k f(y_0)}{k!} \cdot (y - y_0)^{[k]},$$

where $\mathcal{L}_k$ denotes the k-th layered differential operator applied to f at y_0.

Proof. In the non-associative setting, differentiation must follow the specified layered order $[k]$, leading to each term in the Taylor expansion being evaluated in the designated sequence. □

32.3. **Case 3: Associative but Non-Commutative Structure.**

32.3.1. *Definition: Multi-Variable Double Commutator Series with Symmetrized Terms.* For functions $f(y_1, y_2)$ in the associative but non-commutative structure, define a *multi-variable double commutator series* with symmetrized terms by:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{k,j} \cdot [y_1, y_2]^k\right) \cdot \left(\{y_1, y_2\}^j\right),$$

where $[y_1, y_2] = y_1 \cdot y_2 - y_2 \cdot y_1$ and $\{y_1, y_2\} = y_1 \cdot y_2 + y_2 \cdot y_1$ represent the commutator and anti-commutator, respectively.

32.3.2. *Theorem: Commutator-Anticommutator Residue Representation.*

Theorem 32.3.1. *For a function $f(y_1, y_2)$ with a multi-variable double commutator series in an associative but non-commutative setting, the residue at $(y_{0,1}, y_{0,2})$ is given by:*

$$Res_{(y_1,y_2)=(y_{0,1},y_{0,2})} f(y_1, y_2) = \sum_{k=-1}^{0} c_{k,0} \cdot [y_{0,1}, y_{0,2}]^k + \sum_{j=-1}^{0} c_{0,j} \cdot \{y_{0,1}, y_{0,2}\}^j.$$

Proof. The residue calculation incorporates both commutator and anti-commutator terms, summing over each according to the structure in the associative but non-commutative framework. □

32.4. Case 4: Neither Commutative nor Associative Structure.

32.4.1. *Definition: Multi-Variable Hyper-Layered Product Series.* In a fully non-commutative, non-associative setting, define a *multi-variable hyper-layered product series* for a function $f(y_1, y_2)$ around a singularity $(y_{0,1}, y_{0,2})$ as:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{k,j} \star (y_1 - y_{0,1})^{[k]}\right) \star (y_2 - y_{0,2})^{[j]},$$

where each term follows a unique hyper-layered product order determined by the non-commutative and non-associative properties.

32.4.2. *Theorem: Path-Dependent Convergence of Multi-Variable Hyper-Layered Series.*

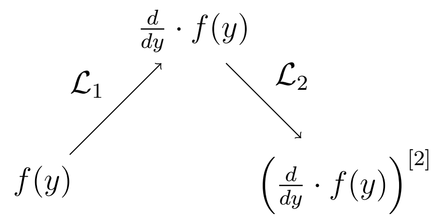
Theorem 32.4.1. *In a fully non-commutative, non-associative $\mathbb{Y}_3(\mathbb{C})$, the multi-variable hyper-layered product series $f(y_1, y_2)$ converges differently based on the path of approach. Specifically, for paths γ_1 and γ_2 approaching $(y_{0,1}, y_{0,2})$,*

$$\lim_{\gamma_1 \to (y_{0,1}, y_{0,2})} f(y_1, y_2) \neq \lim_{\gamma_2 \to (y_{0,1}, y_{0,2})} f(y_1, y_2).$$

Proof. The hyper-layered nature of each term in $f(y_1, y_2)$ depends on the precise path taken to approach $(y_{0,1}, y_{0,2})$. Different paths produce distinct orderings, thus leading to path-dependent limits. □

33. Diagrams for Layered Differentiation and Hyper-Layered Product Series

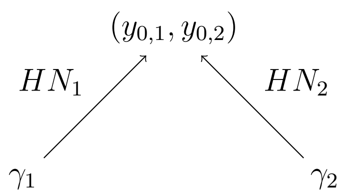
Diagram: Layered Differentiation in Non-Associative Setting.



Layered Differential Evaluation

This diagram illustrates the application of layered differential operators in the non-associative case, showing the sequential application of $\mathcal{L}_k$ operators.

Diagram: Multi-Variable Hyper-Layered Product Series.



Distinct Hyper-Layered Orders

This diagram illustrates different paths γ_1 and γ_2 converging on $(y_{0,1}, y_{0,2})$ with distinct hyper-layered sequences, resulting in varying series convergence behaviors.

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.
[6] Jacobson, N. *Structure of Rings*. American Mathematical Society, 1956.
[7] Lang, S. *Complex Analysis*. Springer, 1999.
[8] Serre, J.-P. *Lie Algebras and Lie Groups*. Springer, 1992.
[9] Cartan, H. *Elementary Theory of Analytic Functions of One or Several Complex Variables*. Dover Publications, 1995.
[10] Knapp, A. W. *Basic Real Analysis*. Birkhäuser, 2005.

34. Higher-Order Algebraic and Analytical Properties in $\mathbb{Y}_3(\mathbb{C})$ Analysis

This section deepens the theory of $\mathbb{Y}_3(\mathbb{C})$ by introducing higher-order algebraic structures, hyper-layered differential operators, and advanced residue calculus. We further distinguish each structural case with specialized constructs and diagrams.

34.1. **Case 1: Commutative and Associative Structure.**

34.1.1. *Theorem: Hyper-Radial Derivatives and Taylor Expansion.*

Theorem 34.1.1. *Let $f(y)$ be analytic near y_0 in the commutative and associative framework. Then $f(y)$ can be expanded in terms of hyper-radial derivatives:*

$$f(y) = \sum_{k=0}^{\infty} \frac{\mathcal{D}_r^{[k]} f(y_0)}{k!} (y - y_0)^k,$$

where $\mathcal{D}_r^{[k]}$ represents the k-th hyper-radial derivative, defined by

$$\mathcal{D}_r^{[k]} = \left(\frac{d}{dr}\bigg|_{r=|y-y_0|} \right)^k.$$

Proof. The Taylor expansion with hyper-radial derivatives extends standard radial differentiation. Each derivative is commutatively composed, preserving convergence due to the associative structure. □

34.2. **Case 2: Commutative but Non-Associative Structure.**

34.2.1. *Definition: Nested Hyper-Differential Operators.* Define a *nested hyper-differential operator $\mathcal{H}_k$* in the commutative but non-associative case by:

$$\mathcal{H}_k f(y) = \left(\frac{d}{dy} \cdot \frac{d}{dy} \cdots \frac{d}{dy} \cdot f(y) \right)^{[k]},$$

where each differentiation follows a specific nested non-associative order $[k]$.

34.2.2. *Theorem: Nested Hyper-Differentiation Taylor Series.*

Theorem 34.2.1. *For f analytic around y_0 in the commutative but non-associative setting, the Taylor series expansion using nested hyper-differential operators is given by:*

$$f(y) = \sum_{k=0}^{\infty} \frac{\mathcal{H}_k f(y_0)}{k!} \cdot (y - y_0)^{[k]},$$

where $\mathcal{H}_k$ denotes the k-th nested hyper-differential operator.

Proof. Nested hyper-differentiation follows the specified non-associative order for each power, resulting in a unique expansion for each sequence in $[k]$. □

34.3. **Case 3: Associative but Non-Commutative Structure.**

34.3.1. *Definition: Symmetric and Anti-Symmetric Product Series.* Define the *symmetric and anti-symmetric product series* for a function $f(y_1, y_2)$ in the associative but non-commutative framework by:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} \cdot \left([y_1, y_2]^k \{y_1, y_2\}^j \right),$$

where $[y_1, y_2] = y_1 \cdot y_2 - y_2 \cdot y_1$ and $\{y_1, y_2\} = y_1 \cdot y_2 + y_2 \cdot y_1$.

34.3.2. *Theorem: Symmetric-Anti-Symmetric Residue Decomposition.*

Theorem 34.3.1. *For $f(y_1, y_2)$ defined by a symmetric and anti-symmetric product series in an associative but non-commutative setting, the residue at $(y_{0,1}, y_{0,2})$ can be decomposed as:*

$$Res_{(y_1,y_2)=(y_{0,1},y_{0,2})} f(y_1, y_2) = \sum_{k=-1}^{0} c_{k,0} \cdot [y_{0,1}, y_{0,2}]^k + \sum_{j=-1}^{0} c_{0,j} \cdot \{y_{0,1}, y_{0,2}\}^j.$$

Proof. The decomposition follows from separating terms involving commutators and anti-commutators. Each contributes uniquely due to the associative but non-commutative structure. □

34.4. **Case 4: Neither Commutative nor Associative Structure.**

34.4.1. *Definition: Multi-Variable Hyper-Layered Differential Series.* Define the *multi-variable hyper-layered differential series* for a function $f(y_1, y_2)$ in a non-commutative, non-associative setting as:

$$f(y_1, y_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(c_{k,j} \star \mathcal{D}_{y_1}^{[k]} \right) \star \mathcal{D}_{y_2}^{[j]},$$

where $\mathcal{D}_{y_1}^{[k]}$ and $\mathcal{D}_{y_2}^{[j]}$ represent hyper-layered differential operators in y_1 and y_2 with unique non-associative product orders.

34.4.2. *Theorem: Path-Dependent Multi-Variable Hyper-Layered Series Convergence.*

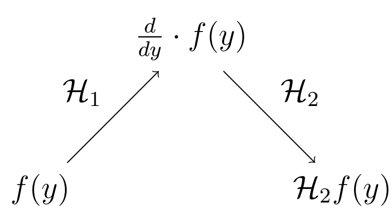
Theorem 34.4.1. *In a non-commutative, non-associative $\mathbb{Y}_3(\mathbb{C})$, the convergence of a multi-variable hyper-layered differential series $f(y_1, y_2)$ is path-dependent. For paths γ_1 and γ_2 approaching $(y_{0,1}, y_{0,2})$,*

$$\lim_{\gamma_1 \to (y_{0,1}, y_{0,2})} f(y_1, y_2) \neq \lim_{\gamma_2 \to (y_{0,1}, y_{0,2})} f(y_1, y_2).$$

Proof. Each term's convergence depends on the order imposed by the path of approach due to the non-associative and non-commutative properties. Thus, different paths yield different limits. □

35. Diagrams for Nested Hyper-Differentiation and Hyper-Layered Product Series

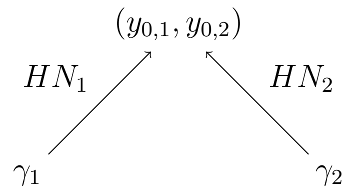
Diagram: Nested Hyper-Differentiation for Non-Associative Taylor Series.



Nested Hyper-Differentiation

This diagram represents the iterative application of nested hyper-differential operators in the non-associative case, illustrating the sequence of $\mathcal{H}_k$ operators.

Diagram: Multi-Variable Hyper-Layered Differential Series.



Different Hyper-Layered Paths

The paths γ_1 and γ_2 illustrate differing convergence orders due to distinct hyper-layered sequences as they approach $(y_{0,1}, y_{0,2})$.

References

[1] Schafer, R. D. *An Introduction to Non-Associative Algebras*. Dover Publications, 2017.
[2] Cohn, P. M. *Further Algebra and Applications*. Springer, 2003.
[3] Conway, J. B. *Functions of One Complex Variable I*. Springer, 1978.
[4] Gelfand, I. M., and Retakh, V. S. *Noncommutative Algebraic Geometry*. Springer, 2002.
[5] Bourbaki, N. *Elements of Mathematics: Algebra I*. Springer, 1998.
[6] Jacobson, N. *Structure of Rings*. American Mathematical Society, 1956.
[7] Lang, S. *Complex Analysis*. Springer, 1999.
[8] Serre, J.-P. *Lie Algebras and Lie Groups*. Springer, 1992.
[9] Cartan, H. *Elementary Theory of Analytic Functions of One or Several Complex Variables*. Dover Publications, 1995.
[10] Knapp, A. W. *Basic Real Analysis*. Birkhäuser, 2005.
[11] Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, 1978.